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Perturbative Heavy Quark Fragmentation Function through $\mathcal{O}(\alpha_s^2)$: Gluon Initiated Contribution

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We derive the gluon initiated contribution to the initial condition for the perturbative fragmentation function of a heavy quark through order $\mathcal{O}(\alpha_s^2)$ in the $\overline{\rm MS}$ scheme. This result is needed for the resummation with next-to-next-to-leading logarithmic accuracy of quasi-collinear logarithms $\ln^k(m^2)$ in heavy quark differential distributions by solving the complete DGLAP equation. Together with the previously evaluated fermion initiated components, this result completes the derivation of the initial condition for the perturbative fragmentation function at next-to-next-to-leading order.

I. INTRODUCTION

Production of heavy flavors (charm and bottom) in high energy processes has become an increasingly important subject in the last decade. In this type of processes, not-completely inclusive observables like energy or transverse momentum distributions of heavy flavored hadrons are of special interest. The reason for that lies in their dependence on both the short-distance perturbative physics and the long-distance physics governing the formation of the heavy flavored hadrons. A peculiar feature of these observables is that they contain logarithmic sensitivity to the mass of the heavy flavor to all orders in perturbation theory. For that reason perturbative calculations of such cross-sections are not reliable when the typical hard scale in the process is much larger than the mass of the produced heavy quark.

A formalism for the study of differential distributions for heavy quark production was proposed long ago by Mele and Nason [1]. It is known as the Perturbative Fragmentation Function (PFF) formalism and its idea is that in order to restore the applicability of perturbative QCD to the type of observables discussed above, one has to resum classes of large logarithms to all orders in perturbation theory. The method is process independent and is based on the DGLAP evolution equation [2].

The main goal of this paper is to derive the gluon initiated component of the initial condition for the perturbative fragmentation function at order $\mathcal{O}(\alpha_s^2)$. Combined with the fermion initiated contributions [3] at order $\mathcal{O}(\alpha_s^2)$, and with the three-loop time-like splitting functions (when the latter become available), this result can be used to perform resummations of the large collinear logs with next-to-next-to-leading logarithmic (NNLL) accuracy.

In general, the importance of the gluon initiated contribution increases in processes with larger separation between the hard scale and the quark mass or in processes where gluon production is not suppressed compared to the production of heavy quarks. This expectation is supported from the previous applications of the PFF formalism at the NLL accuracy level. The effect of the inclusion of the gluon component is certainly important for hadron colliders [4, 5]. The effect of the gluon component in e^+e^- has been discussed, for example in [6]. Therefore, at the NNLL precision level, both the effect of the gluon initiated component and the quark-gluon mixing will be important. In fact, as we demonstrate in this paper, the property of the initial condition for the PFF – that its gluon initiated component at order $\mathcal{O}(\alpha_s)$ vanishes for $\mu_0 = m$, does not persist at order $\mathcal{O}(\alpha_s^2)$.

This paper is organized as follows: in the next section we review some general results and then describe the derivation of the gluon initiated component. In section III we present our result and discuss its properties. At the end we present our conclusions.

II. PROCESS-INDEPENDENT DERIVATION OF $D_g^{\rm ini}$

Let us consider heavy flavor production in a hard scattering process. We are interested in the processes for which the characteristic hard scale Q is much larger than the mass m of the heavy quark. It follows from the QCD factorization theorems [7, 8, 9, 10] that the non-perturbative contributions to the energy distribution of the heavy-flavored hadron

produced in such reaction are contained in the so-called non-perturbative fragmentation function. That function describes the formation of the observed heavy flavored hadron from the heavy quark $\mathcal{Q}$ at a scale set by the mass m, and is typically extracted from e^+e^- data. In this paper we will not be concerned with the non-perturbative component of the fragmentation function; details about its implementation can be found in [1, 4, 5, 6, 11, 12, 13, 14, 15, 16, 17].

We will be only interested in the perturbative production of a heavy quark with mass $m << Q$. In such kinematics all power corrections $\sim (m/Q)^p$ are neglected. This means that at large scales the heavy quark behaves as massless and the role of its mass is, essentially, to regulate the appearing collinear divergences.

For the energy distribution of the heavy quark one writes:

$$\frac{d\sigma_{\mathcal{Q}}}{dz}(z,Q,m) = \sum_a \int_z^1 \frac{dx}{x}\frac{d\hat{\sigma}_a}{dx}(x,Q,\mu) D_{a/\mathcal{Q}}\left(\frac{z}{x},\frac{\mu}{m}\right) + \mathcal{O}\left(\frac{m}{Q}\right)^p . \tag{1}$$

Here m is the mass and $z = E_{\mathcal{Q}}/E_{\mathcal{Q},max}$ the energy fraction of the heavy quark produced in the reaction. The sum in Eq.(1) runs over all partons – quarks, antiquarks (including the flavor $\mathcal{Q}$) and gluons – that can be produced in the hard process and μ is the factorization scale. The function $d\hat{\sigma}_a/dx$, also known as coefficient function, is the collinearly renormalized differential cross-section for producing a massless parton a. The coefficient function is related to the (bare) cross-section $\sigma_a(z,Q,\epsilon)$ for producing a massless parton of type a in the same reaction through the following relation:

$$\frac{d\sigma_a}{dz}(z,Q,\epsilon) = \sum_b \frac{d\hat{\sigma}_b}{dz}(z,Q,\mu) \otimes \Gamma_{ba}(z,\mu,\epsilon). \tag{2}$$

Eq.(2) makes explicit the factorization of the collinear singularities present in the massless cross-section $d\sigma_a/dz(z,Q,\epsilon)$. All IR divergences are regularized by working in $d = 4-2\epsilon$ dimensions. Throughout this paper we will work in the $\overline{\rm MS}$ scheme where the collinear counterterms Γ_{ab} take the form:

$$\Gamma_{ba} = \delta_{ab}\delta(1-z) - \left(\frac{\alpha_s}{2\pi}\right)\frac{P^{(0)}_{ab}(z)}{\epsilon} + \left(\frac{\alpha_s}{2\pi}\right)^2\left[\frac{1}{2\epsilon^2}\left(P^{(0)}_{ac}\otimes P^{(0)}_{cb}(z) + \beta_0 P^{(0)}_{ab}(z)\right) - \frac{1}{2\epsilon}P^{(1)}_{ab}(z)\right]. \tag{3}$$

Above, $\alpha_s = \alpha_s(\mu)$ stands for the strong coupling constant renormalized in the $\overline{\rm MS}$ scheme. Its relation to the bare couplings can be found in [3]. The renormalized coupling evolves with n_f fermion flavors (including $\mathcal{Q}$), $\beta_0 = (11C_A - 4T_R n_f)/6$ is the $\mathcal{O}(\alpha_s^2)$ coefficient of the QCD β-function, $C_A = 3,\ T_R = 1/2$ and $C_F = 4/3$ are the QCD color factors and $P^{(0,1)}_{ab}$ are the time-like splitting functions [18]. Our notations for the splitting functions follow Ref.[10].

The functions $D_{a/\mathcal{Q}}(x,\mu/m)$ in Eq.(1) are the perturbative fragmentation functions [1]. They are solutions of the DGLAP evolution equation and satisfy an initial condition given at some scale $\mu = \mu_0$:

$$D_{a/\mathcal{Q}}\left(z,\frac{\mu_0}{m}\right) = D^{\rm ini}_a\left(z,\frac{\mu_0}{m}\right). \tag{4}$$

Once the factorization scheme for the collinear singularities has been chosen, the initial condition $D^{\rm ini}_a$ is an unique, process independent function that has to be obtained from an additional calculation. A prescription for the evaluation of the initial condition can be extracted from the PFF method. The key observation is that if the initial scale μ_0 is chosen to be of the order of the mass m, then the initial condition can not contain large logarithms and therefore can be derived from fixed order perturbative calculations. In particular, $D^{\rm ini}_a$ can be cast in the following form:

$$D^{\rm ini}_a\left(z,\frac{\mu_0}{m}\right) = \sum_{n=0}\left(\frac{\alpha_s(\mu_0)}{2\pi}\right)^n d^{(n)}_a\left(z,\frac{\mu_0}{m}\right). \tag{5}$$

Note that from the above equations and from the LO result $d^{(0)}_a \sim \delta_{a\mathcal{Q}}$ it also follows that the knowledge of $D^{\rm ini}$ to a given perturbative order allows one to perform $\overline{\rm MS}$ scheme subtraction (at the same order) in calculations where the collinear divergences are regulated with non-zero quark mass (see for example Ref.[19] for more details on that point).

To evaluate the initial condition one needs to combine Eqs.(1) and (2) and truncate Eq.(1) to a fixed perturbative order by identifying the fragmentation function with its initial condition. As a result, $D^{\rm ini}_a$ can be expressed as a convolution of the following perturbative factors: a known universal counterterm Γ_{ab} and two cross-sections $d\sigma_{\mathcal{Q}}$ and $d\hat{\sigma}_a$ that can be separately evaluated in perturbation theory. Using results for heavy and massless quark production in e^+e^-, the functions $d^{(1)}_a$ were first obtained within this approach [1].

Although this method works to any perturbative order, its application beyond next-to-leading (NLO) order is impractical due to the increased complexity in the evaluation of massive cross-sections at higher orders in perturbation theory. A better, process independent method for the calculation of the initial condition $d_a^{(n)}$ was suggested in [11, 20] and further developed in [3] where the functions $d_a^{(2)},\ a = \mathcal{Q}, \overline{\mathcal{Q}}, q, \bar{q}$ were calculated. The method is based on the observation that one needs to compute only a sufficient, process independent part of the cross-sections for massive (respectively massless) quark production. Moreover, this calculation can be formulated in a process independent way. The main purpose of this paper is to evaluate the function $d_g^{(2)}$ using this approach and thus to complete the derivation of the initial condition of the PFF through order $\mathcal{O}(\alpha_s^2)$. With this result and after the three-loop time-like splitting functions become available, one will be able to evaluate the perturbative fragmentation function with next-to-next-to leading logarithmic accuracy. It follows from Eq.(1) that the knowledge of the PFF with such accuracy will make it possible to evaluate heavy quark spectra at NNLO in any process and up to power corrections $\mathcal{O}(m/Q)$ from pure massless calculations and resum large quasi-collinear logs $\ln^k(m^2/Q^2)$ with NNLL accuracy to all orders in the strong coupling constant.

Next, we describe the process independent derivation of the gluon initiated component $d_g^{(2)}$. As was shown in [3], the initial condition for the perturbative fragmentation function can be written as:

$$D_g^{\text{ini}}\left(z, \frac{\mu_0}{m}\right) = \sum_b \Gamma_{gb}(z, \mu_0) \otimes \widetilde{D}_{b/\mathcal{Q}}\left(z, \frac{\mu_0}{m}\right), \tag{6}$$

with Γ_{ab} given in Eq.(3).

The function $\widetilde{D}_{b/\mathcal{Q}}$ can be interpreted as a bare fragmentation function for production of a massive quark $\mathcal{Q}$ in the decay of an (off-shell) parton of type b. That function embodies all the effect of collinear radiation with $q_\perp \sim m \leq \mu_0$ in that process. In general, the collinear limit is defined as the kinematic limit where the relative transverse momentum of two or more particles vanishes. When masses are introduced, the collinear limit is [21]:

$$q_\perp = \kappa q_\perp,\ m = \kappa m,\ \kappa \to 0. \tag{7}$$

The following Sudakov parametrization is particularly convenient for the discussion of the collinear limit in the process $g^* \to \mathcal{Q} + X$ at order $\mathcal{O}(\alpha_s^2)$. Consider the one-to-three collinear splitting of a gluon. We denote the four-momenta of the produced massive quark and antiquark by $q_{1,2}$ while q_3 denotes the momentum of the radiated gluon. The momentum of the decaying off-shell gluon is denoted as $\widehat{p}$:

$$\widehat{p} = q_1 + q_2 + q_3. \tag{8}$$

We parameterize the collinear direction by p and introduce another light-like vector n as the complimentary light-cone vector. We then write:

$$q_i \ = \ z_i p + \beta_i \frac{n}{(pn)} + q_{i,\perp}. \tag{9}$$

The components β_i are found from the on-shell conditions $q_i^2 = m_i^2$:

$$\beta_i = \frac{m_i^2 - q_{i,\perp}^2}{2z_i} \quad , \quad i = 1, 2, 3 \ , \tag{10}$$

with $p^2 = 0$. We are interested in the symmetric case:

$$z_1 + z_2 + z_3 = 1 \ , \quad q_{1,\perp} + q_{2,\perp} + q_{3,\perp} = 0. \tag{11}$$

Using the above equations, we express the momentum $\widehat{p}$ through p and n:

$$\widehat{p} = p + \frac{1}{2}\widehat{p}^2 \frac{n}{(pn)}, \tag{12}$$

where:

$$\widehat{p}^2 = \frac{2(pq_2) + 2(pq_3) - 2(q_2 q_3)}{z}.$$

We often write z instead of z_1 to denote the measured energy fraction of the heavy quark in the final sate. Eqs.(9,11) imply the energy conservation condition:

$$z = 1 - \frac{(nq_2)}{(pn)} - \frac{(nq_3)}{(pn)}. \tag{13}$$

The kinematics of the one-to-two splitting can be obtained by setting the momentum q_3 to zero in Eqs.(8-13).

The explicit expression for the bare fragmentation function appearing in Eq.(6) follows directly from the factorization of phase-space and matrix elements in the collinear kinematics. Consider a hard scattering process characterized by some scale $Q \gg m$. Suppose that $n+2$ partons with momenta $k_1, \ldots, k_{n-1}, q_1, q_2, q_3$ are produced. We consider momenta $k_1, \ldots, k_{n-1}$ as non-exceptional, while momenta q_1, q_2, q_3 are collinear, as described above. We denote the phase-space of $n+2$ partons as $\mathrm{dPS}^{(n+2)}(k_1, \ldots, k_{n-1}, q_1, q_2, q_3)$. In the limit when q_1, q_2 and q_3 become collinear and are resulting from the decay of an off-shell gluon, $q_1 + q_2 + q_3 = p + \mathcal{O}(q_\perp)$, the $(n+2)$-particle phase space factorizes:

$$\mathrm{dPS}^{(n+2)}(k_1, \ldots, k_{n-1}, q_1, q_2, q_3) = \mathrm{dPS}^{(n)}(k_1, \ldots, k_{n-1}, p)\ \frac{1}{z}[dq_2; m][dq_3; 0]. \tag{14}$$

The factor $[dq; m_q]$ in Eq.(14) is the d-dimensional one-particle phase space:

$$[dq; m_q] = \frac{d^d q}{(2\pi)^{d-1}} \delta^+(q^2 - m_q^2). \tag{15}$$

We next discuss the factorization of matrix elements in the collinear kinematical configuration described above. As follows from Eq.(1), we are not interested in power-suppressed contributions to the cross-section. To identify the logarithmically enhanced terms we apply well known power counting arguments [8] in the collinear limit for the scattering amplitudes: we rescale all momenta and masses according to Eq.(7), and then identify the leading contributions of the amplitudes in the limit $\kappa \to 0$. Throughout the paper we work in the light-cone [1] gauge $n_\mu A^\mu = 0$, $n^2 = 0$. This is necessary for the process-independent derivation of D^{ini} because in such gauges non-vanishing contributions to the fragmentation function are produced only by diagrams where collinear radiation is both emitted and absorbed by the same parton. In this paper we are concerned with the case of collinear decay of a gluon. Examples of relevant diagrams are shown on Fig.(1).

For the derivation of the fragmentation function we need to consider only the spin-averaged case [2] . Then, in the collinear limit Eq.(7), matrix element factorization holds in the following form:

$$|M^{(n+2)}(k_1, \ldots, k_{n-1}, q_1, q_2, q_3)|^2 = |M^{(n)}(k_1, \ldots, k_{n-1}, p)|^2\ W(n, \widehat{p}, q_2, q_3) + \mathcal{O}(\kappa). \tag{16}$$

Here $|M^{(n)}(...., p)|^2$ is the squared matrix element for producing n on-shell particles with non-exceptional momenta. All the contribution to the fragmentation function from the collinear splitting of a gluon is contained in the function W. The form of that function in presence of massless fermions is given in [22]:

$$W = -\frac{1}{2(1-\epsilon)} g^{\mu\nu} V_{\mu\nu}. \tag{17}$$

The factor $V_{\mu\nu}$ represents the squared, color averaged sum of diagrams contributing to the decay of an off-shell gluon with momentum given by Eq.(8).

It is easy to generalize this result to the case when the final state quark and antiquark have non-zero mass m. Repeating the considerations in [22] one notices that the inclusion of non-zero masses does not introduce new tensor structure in the tensor decomposition of the function $V_{\mu\nu}$. Then, since the limit $m \to 0$ is regular, the only effect of the inclusion of the mass m is to alter the corresponding scalar form-factors in that tensor decomposition. Finally, we recall that in the collinear limit the mass m is considered to be of the order of the transverse momenta of the collinear particles. Therefore, we conclude that the inclusion of masses does not change the dimensionality of the scalar form-factors or their scaling in the collinear limit, i.e. in the presence of non-zero quark masses the function W is also given by Eq.(17).

Combining Eqs.(1,13,14,16) we find that the function $\widetilde{D}_{g/\mathcal{Q}}(z)$ appearing in Eq.(6) can be written as:

$$\widetilde{D}_{g/\mathcal{Q}}(z) = \frac{1}{z} \int\ W\ [dq_2; m][dq_3; 0] \delta\left(1 - z - \frac{(nq_2)}{(pn)} - \frac{(nq_3)}{(pn)}\right). \tag{18}$$

Similar expression holds for the one-loop virtual contribution to that function. In this case one should replace in Eq.(18) the phase-space factor $[dq_3; 0]$ with the corresponding phase-space for the virtual integration and omit the term with q_3 in the energy-conservation delta-function. In both cases the function W is evaluated from Eq.(17) and the tensor $V_{\mu\nu}$ is constructed from the appropriate diagrams. Two examples of such diagrams are shown on Fig.(1).

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[1] For simplicity, the gauge fixing vector n is chosen to be the same vector that appears in the Sudakov parametrization Eq.(9).

[2] In general, in the case of decaying gluon the matrix elements do not factorize due to the presence of non-trivial spin correlations [22, 23].

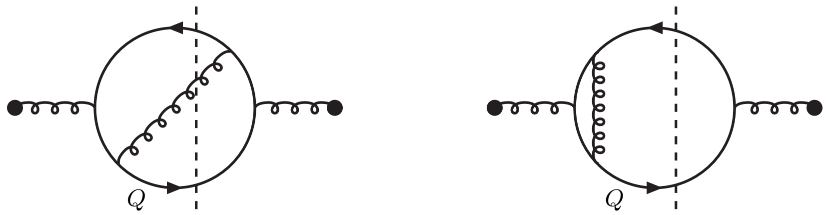

FIG. 1: Examples of diagrams for the gluon decay process $g \to Q + X$ at $\mathcal{O}(\alpha_s^2)$ that contribute to the function W. The dashed vertical line indicates the intermediate state that has to be considered.

III. RESULTS

The first step in the evaluation of $\widetilde{D}_{g/\mathcal{Q}}(z)$ is to obtain the splitting function W in Eq.(17). Our calculation generalizes the tree-level massless results in [22, 23]. The next step is the integration over the corresponding real and/or virtual momenta. To perform both the phase-space and loop integrals, we follow the methods described in [24, 25, 26]: we first present all non-trivial phase-space integrals as loop integrals and then use standard multiloop methods [27], such as integration-by-parts and recurrence relations, to reduce all the phase-space and loop integrals that have to be evaluated to a set of 18 master integrals. For the reduction to master integrals we use the algorithm [28] implemented in [29]. All algebraic manipulations have been performed using Maple [30] and Form [31].

The coupling renormalization requires the result for the function $\widetilde{D}_{g/\mathcal{Q}}(z)$ at order α_s including terms of order $\mathcal{O}(\epsilon)$. The corresponding result can be found in [3]. Field renormalization is performed by calculating one-loop self-energy insertions in the external quark legs. The use of light-cone gauge does not lead to appearance of singularities that are not regulated by dimensional regularization. As a last step we perform the collinear renormalization according to Eqs.(3) and (6).

We decompose the result for $d_g^{(2)}(z, \mu_0/m)$ in terms of the color factors:

$$d_g^{(2)}\left(z, \frac{\mu_0}{m}\right) = C_F T_R F_g^{(C_F T_R)} + C_A T_R F_g^{(C_A T_R)} + T_R^2 F_g^{(T_R^2)} + T_R^2 n_l F_g^{(T_R^2 n_l)}, \tag{19}$$

where $n_l = n_f - 1$ is the number of massless flavors. The functions F_g read:

$$\begin{aligned}
F_g^{(C_F T_R)} = {} & \Big\{ -\frac{4z^2-2z+1}{2}\ln(z) + z - \frac{1}{4} + (2z^2-2z+1)\ln(1-z)\Big\} L^2 + \Big\{ 2(4z^2-6z+3)\mathrm{Li}_2(z) \\
& -\frac{4z^2-2z+1}{2}\ln^2(z) + 6(2z^2-2z+1)\ln(1-z)\ln(z) - 3(2z^2-2z+1)\ln^2(1-z) + \frac{8z-3}{2}\ln(z) \\
& -(4z+1)\ln(1-z) - \frac{10z^2-13z+8}{2} + \frac{2\pi^2 z^2}{3}\Big\} L - 16(2z^2-2z+1)\left(\mathrm{Li}_3\left(\frac{2z-1}{z}\right) + \mathrm{Li}_3\left(\frac{2z-1}{z-1}\right)\right) \\
& -2(8z^2-10z+5)\mathrm{Li}_3(z) - 16(2z^2-2z+1)\mathrm{Li}_3(1-z) + \Big\{ -8(2z^2-2z+1)\ln(z) + 8(2z^2-2z+1)\ln(1-z) \\
& +\frac{8(1-2z)^3}{3}\Big\}\mathrm{Li}_2\left(\frac{2z-1}{z}\right) + \left(-12(2z^2-2z+1)\ln(1-z) + (8z^2-10z+5)\ln(z) + \frac{32z^3-24z^2+12z-1}{6}\right) \\
& \times \mathrm{Li}_2(z) - \frac{36z^2-34z+17}{12}\ln^3(z) + \frac{5(2z^2-2z+1)}{2}\ln^3(1-z) - \frac{29(2z^2-2z+1)}{2}\ln(z)\ln^2(1-z) \\
& +\frac{11(2z^2-2z+1)}{2}\ln^2(z)\ln(1-z) - \frac{256z^3-372z^2+156z-29}{24}\ln^2(z) - \frac{32z^3-46z^2+22z-5}{4}\ln^2(1-z) \\
& +\frac{64z^3-87z^2+33z+5}{3}\ln(z)\ln(1-z) + \left(\frac{96z^3-196z^2+255z-48}{12} - (2z^2-2z+1)\frac{\pi^2}{6}\right)\ln(z) \\
& -\left(\frac{24z^3-65z^2+79z-26}{3} - (2z^2-2z+1)\frac{11\pi^2}{6}\right)\ln(1-z) - 2\pi^2(2z^2-2z+1)\big|\ln(z)-\ln(1-z)\big| \\
& +\frac{2\pi^2}{3}\big|1-2z\big|(1-2z)^2 + (32z^3-54z^2+6z+19)\frac{\pi^2}{36} + (56z^2-60z+30)\zeta(3) - \frac{152z^2-173z+69}{12} \quad ,
\end{aligned} \tag{20}$$

$$
\begin{aligned}
F_g^{(C_A T_R)} &= \left\{(4z+1)\ln(z)+(2z^2-2z+1)\ln(1-z)-\frac{9z^2-2z-14}{6}+\frac{2}{3z}\right\}L^2+\left\{-8(2z^2-2z+1)\mathrm{Li}_2(z)\right.\\
&-2(2z^2+2z+1)\mathrm{Li}_2(-z)+(6z+1)\ln^2(z)-4(2z^2-2z+1)\ln(z)\ln(1-z)-2(2z^2+2z+1)\ln(z)\ln(1+z)\\
&+(2z^2-2z+1)\ln^2(1-z)-\frac{2(2z^2+17z+2)}{3}\ln(z)+\frac{10z^2-10z+11}{3}\ln(1-z)-\frac{2\pi^2 z}{3}+\frac{178z^2-95z+13}{9}\\
&\left.-\frac{20}{9z}\right\}L+2(2z^2+2z+1)\left[S_{1,2}(-z)-S_{1,2}(z^2)\right]+8(2z^2-2z+1)\left(\mathrm{Li}_3\left(\frac{2z-1}{z}\right)+\mathrm{Li}_3\left(\frac{2z-1}{z-1}\right)\right)\\
&+(6z^2+18z+7)\mathrm{Li}_3(z)+2(2z^2-10z+1)\mathrm{Li}_3(1-z)+(2z^2+2z+1)\mathrm{Li}_3(-z)+\left\{4(2z^2-2z+1)\ln(z)\right.\\
&\left.-4(2z^2-2z+1)\ln(1-z)-\frac{4(1-2z)^3}{3}\right\}\mathrm{Li}_2\left(\frac{2z-1}{z}\right)+\left\{-(6z^2+2z+5)\ln(z)+2(2z^2-10z+1)\ln(1-z)\right.\\
&\left.-4(2z^2+2z+1)\ln(1+z)-\frac{8z^3+4z^2-33z-7}{3}+\frac{4}{3z}\right\}\mathrm{Li}_2(z)+\left\{-(2z^2+2z+1)\ln(z)-2(2z^2+2z+1)\right.\\
&\left.\times\ln(1+z)+2z(1+z)\right\}\mathrm{Li}_2(-z)+\frac{8z^2+2z+5}{6}\ln^3(z)-\frac{7(2z^2-2z+1)}{6}\ln^3(1-z)+(10z^2-18z+5)\\
&\times\ln(z)\ln^2(1-z)-3(2z^2-2z+1)\ln^2(z)\ln(1-z)-\frac{2z^2+2z+1}{2}\ln^2(z)\ln(1+z)-(2z^2+2z+1)\\
&\times\ln(z)\ln^2(1+z)+\frac{32z^3-50z^2+25z-6}{6}\ln^2(z)-\left(\frac{64z^3-44z^2+10z-15}{6}-\frac{4}{3z}\right)\ln(z)\ln(1-z)\\
&+2z(1+z)\ln(z)\ln(1+z)+\frac{48z^3-44z^2-16z+23}{12}\ln^2(1-z)-\left(\frac{24z^3-624z^2-475z-181}{18}\right.\\
&\left.+(2z^2+6z+1)\frac{\pi^2}{6}\right)\ln(z)+\left((22z-6z^2-3)\frac{\pi^2}{6}+\frac{12z^3+76z^2-28z-22}{9}\right)\ln(1-z)\\
&+\frac{\pi^2}{2}(2z^2+2z+1)\ln(1+z)+\pi^2(2z^2-2z+1)\big|\ln(z)-\ln(1-z)\big|-\frac{\pi^2}{3}\big|1-2z\big|(1-2z)^2\\
&-(16z^3+24z^2+12z+29)\frac{\pi^2}{36}-2(10z^2-3z+7)\zeta(3)-\frac{2568z^2-1583z-1331}{54}+\frac{56}{27z}\quad,
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
F_g^{(T_R^2)} &= -\frac{2(2z^2-2z+1)}{3}L^2+\left\{-\frac{4(2z^2-2z+1)}{3}(\ln(z)+\ln(1-z))-\frac{4(4z^2-4z+5)}{9}\right\}L\\
&-\frac{2z^2-2z+1}{3}(\ln^2(z)+\ln^2(1-z))+2(2z^2-2z+1)\ln(z)\ln(1-z)\\
&-\frac{128z^5-320z^4+160z^3+40z^2-160z+50}{45}\ln(z)+\frac{128z^5-320z^4+160z^3+120z^2-240z+102}{45}\ln(1-z)\\
&+(2z^2-2z+1)\frac{\pi^2}{3}+\frac{384z^4-768z^3+568z^2-184z-280}{135}\quad,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
F_g^{(T_R^2 n_l)} &= -\frac{2(2z^2-2z+1)}{3}L^2+\left\{-\frac{4(2z^2-2z+1)}{3}(\ln(z)+\ln(1-z))-\frac{4(4z^2-4z+5)}{9}\right\}L-\frac{2z^2-2z+1}{3}\\
&\times[\ln(z)+\ln(1-z)]^2-\frac{2(4z^2-4z+5)}{9}[\ln(z)+\ln(1-z)]+(2z^2-2z+1)\frac{\pi^2}{3}-\frac{88z^2-88z+56}{27}\quad,
\end{aligned}
\tag{23}
$$

where $L=\ln(\mu_0^2/m^2)$.

Eq.(19) is the main result of this paper.

We next comment on the properties of Eq.(19). It contains polylogarithmic functions up to rank three. These functions are defined through:

$$\mathrm{Li}_n(z) = \int_0^z \frac{\mathrm{Li}_{n-1}(x)}{x}dx, \quad \mathrm{Li}_1(z) = -\ln(1-z), \quad S_{1,2}(z) = \frac{1}{2}\int_0^z \frac{\ln^2(1-x)}{x}dx. \tag{24}$$

The mass-enhanced terms proportional to $\ln^k(\mu_0^2/m^2)$, $k = 1, 2$ are of the general form discussed in [3]. The behavior of Eq.(19) in the limits $z \to 0$ and $z \to 1$ is as follows: in the limit $z \to 1$ the gluon initiated initial condition is only as singular as $\sim \ln^3(1-z)$ and therefore suppressed with an inverse power of N for large values of the Mellin variable. For small z, the function $d_g^{(2)}$ exhibits $\sim C_A T_R/z$ pole and, in addition, has less singular terms $\sim \ln^n(z)$, $n \leq 3$.

There are two more terms in $d_g^{(2)}$ that deserve comment. They contain the functions $|\ln(z) - \ln(1-z)|$ and $|1-2z|$. Those terms are proportional to the color factor $(C_F - C_A/2)T_R$ and originate from the virtual diagram representing the quark-gluon vertex correction shown on Fig.(1). To interpret these terms we first rewrite them as $2\mathrm{arctanh}(\sqrt{(1-2z)^2})$ and $\sqrt{(1-2z)^2}$ respectively. We see that in the point $z = 1/2$ these functions are not smooth.

At the point $z = 1/2$ the virtual diagram in question represents a configuration where a quark-antiquark pair is created and the both final-state particles have the same energy. Since their relative transverse momentum is small, that configuration represents effectively a threshold production of a heavy pair [32]. It is well known (see [33] for example) that the one-loop vertex correction for heavy pair creation close to threshold contains powers β^n , $n \geq -1$ of the velocity:

$$\beta = \sqrt{1 - \frac{4m^2}{Q^2}}, \tag{25}$$

where m is the mass of the quark and Q the invariant mass of the pair.

One can easily verify that in our collinear kinematics the analogue of β takes the form:

$$\sqrt{1 - \frac{4z(1-z)}{1 + |q_\perp^2|/m^2}}, \tag{26}$$

and for vanishing relative transverse momentum $q_\perp$ between the two final-state particles the threshold condition is indeed $z = 1/2$. With an explicit calculation one can also check that terms from the virtual diagram that are of the form $\gamma^\mu \beta^{2n-1}$, $n \geq 0$, indeed lead to contributions to $d_g^{(2)}$ that are proportional to the functions $\mathrm{arctanh}(\sqrt{(1-2z)^2})$ and $\sqrt{(1-2z)^2}$.

IV. CONCLUSIONS

In this paper we compute the gluon initiated contribution to the initial condition for the perturbative fragmentation function of a heavy quark through order $\mathcal{O}(\alpha_s^2)$ in the $\overline{\mathrm{MS}}$ scheme. To derive this result we follow the general approach developed in [3]. The result has the following properties: the terms with mass logarithms are of the general form predicted by the DGLAP equation. We find that the gluon initiated contribution is suppressed with an inverse power of N in the limit $N \to \infty$, where N is the conjugated to z Mellin variable. The result also exhibits a non-smooth behavior at the mid-point $z = 1/2$ which we interpret as due to Coulomb interaction at threshold. Unlike the order $\mathcal{O}(\alpha_s)$ result, the gluon initiated initial condition at order $\mathcal{O}(\alpha_s^2)$ does not vanish when $m = \mu_0$. Our calculation shows the presence of non-trivial constant term.

Combined with the results for the fermion-initiated contributions derived in [3] this result completes the calculation of the initial condition for the perturbative fragmentation function through order $\mathcal{O}(\alpha_s^2)$. After the three-loop time-like splitting functions become available, our results will permit the resummation with next-to-next-to-leading logarithmic (NNLL) accuracy of the large quasi-collinear logs $\sim \ln^k(m^2)$ that appear in heavy quark differential distributions, i.e. will extend the Perturbative Fragmentation Function formalism [1] to the NNLL level.

Our result has large number of potential applications due to the process independence of the PFF method. Of central importance will be its application to e^+e^- in order to extract with NNLL accuracy the non-perturbative fragmentation components. With such result at hand, one can make high precision predictions for heavy flavor energy and transverse momentum distributions at hadron colliders, top-decay, DIS, etc. Combined with the previously calculated fermion initiated components, this result can also be used to evaluate spectra of heavy quarks at NNLO in any process and up to power corrections $(m/Q)^n$, from purely massless calculations.

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